# Pattern Recognition

Hertentamen II , August 30, 2006

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The problems are to be solved within 3 hrs. **The use of supporting material (books, notes) is not allowed.** A calculator may be used, but is not required. In each of the five problems you can achieve up to 2 points, with a total maximum of 10 points. The exam is "passed" with 5.5 or more points.

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## 1. Decision boundaries

a) Explain the term "overfitting of a decision boundary", draw a sketch of a simple example in the context of classification (not regression) in a two–dimensional feature space.

b) Consider the following sets of feature vectors, representing
class 1: $S_1 = \{(2,6),(3,4),(3,8),(4,6)\}$ and
class 2: $S_2 = \{(3,0),(3,-4),(1,-2),(5,-2)\}$, respectively.
They originate from two two-dimensional normal distributions. Compute the covariance matrices for each class from the sample data and write down the corresponding bivariate normal densities. Use naive, biased Maximum Likelihood estimates, here.

c) Assuming equal prior probabilities, evaluate the optimal decision boundary between the classes based on the densities obtained in part b).

## 2. Minimum error rate classification

Consider a simple, binary classification problem which is based on a single feature $x$. Assume that the corresponding class conditional probabilities are

$$p(x|\omega_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-2)^2\right] \quad \text{and} \quad p(x|\omega_2) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-8)^2\right].$$

The classifier decides for $\omega_1$ if $x < x^*$ and else decides for $\omega_2$.

**a)** Which value of the decision boundary $x^*$ gives the lowest expected classification error if the prior probabilities are $P(\omega_1) = P(\omega_2) = 1/2$ ? Visualize the situation, i.e. sketch the class conditionals and mark $x^*$.

**b)** Assume the value $x^*$ from part a) is used, although the true priors are $P(\omega_1) > 1/2$ and $P(\omega_2) = 1 - P(\omega_1)$. Does the expected classification error increase, decrease, or remain the same in comparison with the case $P(\omega_1) = 1/2$?

**c)** Is the <u>optimal</u> boundary for $P(\omega_1) > 1/2$ greater or smaller then $x^*$ for $P(\omega_1) = 1/2$?

Remarks:
Explicit calculations are not necessary here. You can exploit symmetries and use plausibility arguments, instead. However, it is not sufficient to "guess" the correct results, explain your answers!

## 3. Density estimation

a) Define and explain Maximum Likelihood (ML) estimation in the context of density estimation.

b) What are the ML estimates of mean and variance in case of a unidimensional normal distribution as obtained from sample data $\{x_1, x_2, \ldots, x_n\}$? (Just write down the estimates, you don't have to show that they maximize the likelihood.)

c) The ML estimate of the variance is a so-called *biased estimate.* Explain precisely what this means (you don't have to prove that the estimate is biased). Write down an alternative, unbiased estimate of the variance.

## 4. Kullback–Leibler divergence

An important measure of the difference between two distributions in the same space is the so-called *Kullback–Leibler (KL) divergence.* For two densities $p_1(x)$ and $p_2(x)$ (real random number $x$) it is defined as

$$D_{KL}\left[p_1(x), p_2(x)\right] = \int_{-\infty}^{\infty} p_1(x) \ln\left(\frac{p_1(x)}{p_2(x)}\right) dx$$

a) Suppose we want to approximate an arbitrary distribution $p_1(x)$ by a normal density $p_2 = N(\mu, \sigma^2)$ with adjustable mean value $\mu$ and adjustable variance $\sigma^2$. Show that the "obvious" choice

$$\mu = \epsilon_1[x] \quad \text{and} \quad \sigma^2 = \epsilon_1[(x-\mu)^2]$$

satisfies the necessary conditions for minimizing the KL divergence. Here, $\epsilon_1$ denotes the expectation over $p_1$.

b) One can show that the KL divergence is non-negative (you don't have to show it). Hence, it is sometimes called the *KL distance.* Explain why this "distance" is <u>not</u> a metric in the space of distributions $p(x)$. It is sufficient to argue that <u>one</u> of the properties of metrics is violated.

## 5. K–Means algorithm

a) What is the purpose of the *K–Means algorithm*? Present the algorithm in terms of a "pseudocode computer program" and sketch an example scenario for a two–dimensional feature space.

b) What is the essential difference between the *K–Means algorithm* and the *Fuzzy K–Means algorithm* (in words, no mathematical definition of the alg. required) ? What is, supposedly, the advantage of *Fuzzy K–Means*?